Notes 1. CONVEX FUNCTIONS

A function f defined on an interval I is called a **convex function** if it satisfies

$$f((1-\lambda)x+\lambda y) \le (1-\lambda)f(x)+\lambda f(y),\ \forall x,y\in I, \lambda\in[0,1].$$

Observe that $z=(1-\lambda)x+\lambda y$ is a point on the line segment connecting x and y. As λ increases from 0 to 1, z runs from x to y. The line segment in $\mathbb{R}^2$ connecting $(x, f(x))$ and $(y, f(y))$ is given by the graph of the linear function

$$\begin{aligned} l(z) &= \Big(\frac{f(y)-f(x)}{y-x}\Big)(z-x)+f(x) \\ &= \Big(\frac{f(x)-f(y)}{x-y}\Big)(z-y)+f(y). \end{aligned}$$

It is readily checked that f is convex if and only if

$$f(z) \le l(z),$$

for any z lying between x and y. (Here l depends on x and y). This condition has a clear geometric meaning. Namely, the line segment connecting $(x, f(x))$ and $(y, f(y))$ always lies above the graph of f over the interval with endpoints x and y.

A function is called **concave** if its negative is convex. Apparently every result for convex functions has a corresponding one for concave functions. In some situations the use of concavity is more appropriate than convexity.

Proposition 1.1. *Let f be defined on the interval I. For $x,y,z\in I, x<z<y$, f is convex if and only if either one of the following inequalities holds*

$$\frac{f(z)-f(x)}{z-x} \le \frac{f(y)-f(x)}{y-x}, \tag{1.1}$$

$$\frac{f(y)-f(x)}{y-x} \le \frac{f(y)-f(z)}{y-z}. \tag{1.2}$$

Proof. Let $x<y$ be in I. Now f is convex if and only if for $z\in[x,y]$, $f(z)\le l(z)$, that is,

$$f(z) \le \frac{f(y)-f(x)}{y-x}(z-x)+f(x)\ .$$

Move $f(x)$ to the left hand side and then divide both sides by $z-x$ we get (1.1). Similarly, using the second form of $l(z)$ we have

$$f(z) \le \frac{f(x)-f(y)}{x-y}(z-y)+f(y)\ ,$$

so (1,2) follows by first moving $f(y)$ to left and then dividing by $z-y$. $\square$

Geometrically this is evident. We fix x first and consider the point z moving from x to y, (1.1) tells us that the slope keeps increasing. On the other hand, we fix y and consider the point z moving from x to y, (1.2) tells us that again the slope increases.

Proposition 1.2. *Let f be defined on I. Then f is convex if and only if for $x<z<y$ in I,*

$$\frac{f(z)-f(x)}{z-x} \le \frac{f(y)-f(z)}{y-z}.$$

Proof. This inequality can be rewritten as

$$f(z)(y-z)-f(x)(y-z) \le f(y)(z-x)-f(z)(z-x) \ ,$$

which is the same as

$$\begin{aligned} f(z)(y-x) &\le f(y)(z-x)+f(x)(y-z) \\ &= (f(y)-f(x))(z-x)+f(x)(y-x) \ . \end{aligned}$$

Now (1.1) follows by dividing both sides by $y-x$. By Proposition 1 f is convex. We can reverse the reasoning to get the converse. $\square$

Theorem 1.3. *Every convex function f on the open interval I has right and left derivatives, and they satisfy*

$$f'_-(x) \le f'_+(x), \ \forall x \in I, \tag{1.3}$$

and

$$f'_+(x) \le f'_-(y), \ \forall x<y \text{ in } I. \tag{1.4}$$

In particular, f is continuous in I.

We note that f is right continuous at x if $f^+(x)$ exists and is left continuous at x if $f^-(x)$ exists, see the Lemma 1.5 below. Hence it is continuous at x if both one-sided derivatives exist at x. We point out that this theorem does not necessarily hold on a closed interval. For instance, let f be a continuous convex function on $[a,b]$ and define another function g which is equal to f on (a,b), but assign its values at the endpoints so that $g(a)>f(a)$ and $g(b)>f(a)$. Then g is convex on $[a,b]$ but not continuous at a,b.

Proof. From Proposition 1.1 and Proposition 1.2 the function

$$\varphi(t)=\frac{f(t)-f(x)}{t-x}, \ t>x,$$

is increasing and is bounded below by $(f(x) - f(x_0))/(x - x_0)$, where x_0 is any fixed point in I satisfying $x_0 < x$. It follows that $\lim_{t\to x^+} \varphi(t)$ exists. (If you are not sure why this is true, see the Lemma 1.4.) In other words, $f'_+(x)$ exists. Notice that we still have

$$f'_+(x) \geq \frac{f(x) - f(x_0)}{x - x_0},$$

after passing to limit. As the quotient in the right hand side is increasing as x_0 increases to x, by (1.2), we conclude that

$$\lim_{x_0 \to x^-} \frac{f(x) - f(x_0)}{x - x_0} = f'_-(x)$$

exists and (1.3)

$$f'_+(x) \geq f'_-(x)$$

holds. After proving that the right and left derivatives of f exist everywhere in I, we let $z \to x^+$ in (1.1) to get

$$f'_+(x) \leq \frac{f(y) - f(x)}{y - x};$$

and let $z \to y^-$ in (1.2) to get

$$\frac{f(y) - f(x)}{y - x} \leq f'_-(y),$$

whence (1.4) follows.

□

Lemma 1.4. *Let h be an increasing function on (a, b). Suppose that $h(t) \geq -M$, $\forall t \in (a, b)$, for some constant M. Then $\lim_{t\to a^+} h(t)$ exists.*

Proof. We fix a sequence $\{t_n\}$ in (a, b) satisfying $t_n \to a^+$. Since h is increasing and $h \geq -M$, $\{h(t_n)\}$ is a decreasing sequence bounded from below, so $A = \lim_{n\to\infty} h(t_n)$ must exist. For each $\varepsilon > 0$, there is some n_0 such that $0 \leq h(t_n) - A < \varepsilon$ for all $n \geq n_0$. Therefore, for all $t < t_{n_0}$, $h(t) - A \leq h(t_{n_0}) - A < \varepsilon$. On the other hand, since $t_n \to a^+$, we can find some n_1 such that $h(t_{n_1}) \leq h(t)$. Thus, $0 \leq h(t_{n_1}) - A \leq h(t) - A$. By taking $\delta = t_{n_0} - a$, we have $0 \leq h(t) - A < \varepsilon$ for all $t \in (a, a + \delta)$. □

Lemma 1.5. *Let f be a function on (a, b) and $c \in (a, b)$. Then f is right continuous (resp. left continuous) at c if $f'_+(c)$ (resp. $f'_-(c)$) exists. Hence conclude that f is continuous at c if both one-sided derivatives exist at c.*

Proof. Assume $f'_+(c)$ exists. Taking $\varepsilon = 1$, there exists some δ such that

$$\left| \frac{f(x) - f(c)}{x - c} - f'_+(c) \right| < 1 \ , \quad \forall x \in (c, c + \delta) \ .$$

It follows that

$$(f'_+(c) - 1)(x - c) < f(x) - f(c) < (f'_+(c) + 1)(x - c) \ , \forall x \in (c, c + \delta) \ .$$

Hence

$$\lim_{x \to c^+} (f'_+(c) + 1)(x - c) \le \lim_{x \to c^+} (f(x) - f(c)) \le \lim_{x \to c_+} (f'_+(c) + 1)(x - c) \ ,$$

which forces that

$$\lim_{x \to c^+} (f(x) - f(c)) = 0 \ .$$

The other case can be treated similarly. □

The following far-reaching theorem is for optional reading.

Theorem 1.6. ** Every convex function on I is differentiable except possibly at a countable set.*

Proof. Noting that every interval I can be written as the union of countably many closed and bounded intervals, it suffices to show there are at most countably many non-differentiable points in any closed and bounded interval $[a, b]$ strictly contained inside I. Fix a small $\delta > 0$ so that $[a-\delta, b+\delta] \subset I$. Since f is continuous in $[a - \delta, b + \delta]$, it is bounded in $[a - \delta, b + \delta]$. Let $M \ge |f(x)|, \forall x \in [a - \delta, b + \delta]$. By convexity

$$f'_+(b) \le \frac{f(b + \delta) - f(b)}{(b + \delta) - b} \le \frac{2M}{\delta},$$

and

$$f'_-(a) \ge \frac{f(a) - f(a - \delta)}{a - (a - \delta)} \ge \frac{-2M}{\delta},$$

As a result, for $x \in [a, b]$,

$$f'_-(a) \le f'_\pm(x) \le f'_+(b),$$

and the estimate

$$\frac{-2M}{\delta} \le f'_\pm(x) \le \frac{2M}{\delta}.$$

holds. Non-differentiable points in $[a, b]$ belong to the set

$$D = \{x : f'_+(x) - f'_-(x) > 0\} = \bigcup_{k=1}^{\infty} D_k,$$

where $D_k = \{x : f'_+(x) - f'_-(x) \geq \frac{1}{k}\}$. We claim that each D_k is a finite set. To see this let us pick n many points from $D_k : x_1 < x_2 < ... < x_n$. Then

$$\begin{aligned}
& f'_+(x_n) - f'_-(x_1) \\
= \ & \big(f'_+(x_n) - f'_-(x_n)\big) + \big(f'_-(x_n) - f'_-(x_{n-1})\big) + \big(f'_-(x_{n-1}) - f'_-(x_{n-2})\big) + \\
& \cdots + \big(f'_-(x_2) - f'_-(x_1)\big) \\
\geq \ & \big(f'_+(x_n) - f'_-(x_n)\big) + \big(f'_+(x_{n-1}) - f'_-(x_{n-1})\big) + \big(f'_+(x_{n-2}) - f'_-(x_{n-2})\big) + \\
& \cdots + \big(f'_+(x_1) - f'_-(x_1)\big) \\
\geq \ & \frac{n}{k} \ ,
\end{aligned}$$

which imposes a bound on n: $n \leq 4kM/\delta$. □

When f is differentiable, Theorem 1.3 asserts that $f^{'}$ is increasing. The converse is also true.

Theorem 1.7. *Let f be differentiable in I. It is convex if and only if $f^{'}$ is increasing.*

Proof. Theorem 1.3 asserts that f' is increasing if f is convex and differentiable. To show that converse, let $z = (1-\lambda)x + \lambda y \in [x, y]$. Applying the mean-value theorem to f there exist $c_1 \in (x, z)$ and $c_2 \in (z, y)$ such that

$$f(z) = f(x) + f^{'}(c_1)(z - x),$$

and

$$f(y) = f(z) + f^{'}(c_2)(y - z).$$

Using $f^{'}(c_1) \leq f^{'}(c_2)$ we get

$$\frac{f(z) - f(x)}{z - x} \leq \frac{f(y) - f(z)}{y - z},$$

which, by Proposition 1.2, implies that f is convex. □

Theorem 1.8. *Let f be twice differentiable in I. It is convex if and only if $f^{''} \geq 0$.*

Proof. When f is convex, $f^{'}$ is increasing and so $f^{''} \geq 0$. On the other hand, $f^{''} \geq 0$ implies that $f^{'}$ is increasing and hence convex.

□

A function is **strictly convex** on I if it is convex and

$$f((1-\lambda)x + \lambda y) < (1-\lambda)f(x) + \lambda f(y), \quad \forall x < y, \ \lambda \in (0, 1).$$

From the proofs of the above two theorems we readily deduce the following proposition. Likewise, a function is **strictly concave** if its negative is strictly convex.

Proposition 1.9. *The function f is strictly convex on I provided one of the followings hold:*

(a) f is differentiable and f' is strictly increasing; or

(b) f is twice differentiable and $f'' > 0$.

By this proposition, one can verify easily that the following functions are strictly convex.

- $e^{\alpha x}$ where $\alpha \neq 0$ on $(-\infty, \infty)$,
- x^p where $p > 1$ or $p < 0$ on $(0, \infty)$.
- $-\log x$ on $(0, \infty)$.

Convexity is a breeding ground for inequalities. We establish a fundamental one here.

Theorem 1.10 (**Jensen's Inequality**)**.** *For a convex function f on the interval I, let $x_1, x_2, \cdots, x_n \in I$ and $\lambda_1, \lambda_2, \cdots, \lambda_n \in (0, 1)$ satisfying $\sum_{j=1}^n \lambda_j = 1$. Then*

$$f(\lambda_1 x_1 + \cdots + \lambda_n x_n) \leq \lambda_1 f(x_1) + \cdots + \lambda_n f(x_n).$$

When f is strictly convex, equality sign in this inequality holds if and only if $x_1 = x_2 = \cdots = x_n$.

Perhaps we need to explain why the linear combination is still contained in the same interval. WLOG let $x_1 \leq x_2 \leq \cdots \leq x_n$. Then

$$\sum_j \lambda_j x_j \leq \sum_j \lambda_j x_n = x_n \ , \quad \sum_j \lambda_j x_j \geq \sum_j \lambda_j x_1 \geq x_1 \ ,$$

together imply that $\sum_j \lambda_j x_j$ is bounded between x_1 and x_n and hence belongs to I.

Many well-known inequalities including the AM-GM inequality and Hölder inequality are special cases of Jensen's inequality. Some of them are found in the exercise.

Proof. We prove Jensen's inequality by an inductive argument on the number of points. When $n = 2$, the inequality follows from the definition of convexity.

Assuming that it is true for $n-1$ many points, we show its validity for n many points. Let $\lambda_1, \cdots, \lambda_n \in (0,1), \sum_j \lambda_j = 1$ and let

$$y = \sum_{j=1}^{n-1} \frac{\lambda_j}{1-\lambda_n} x_j \ .$$

Using first the definition of convexity and then the induction hypothesis,

$$\begin{aligned} f(\lambda_1 x_1 + \cdots + \lambda_n x_n) &= f((1-\lambda_n)y + \lambda_n x_n) \\ &\le (1-\lambda_n) f(y) + \lambda_n f(x_n) \\ &= (1-\lambda_n) f\left(\sum_{j=1}^{n-1} \frac{\lambda_j}{1-\lambda_n} x_j \right) + \lambda_n f(x_n) \\ &\le (1-\lambda_n) \sum_{j=1}^{n-1} \frac{\lambda_j}{1-\lambda_n} f(x_j) + \lambda_n f(x_n) \\ &= \sum_{j=1}^{n} \lambda_j f(x_j) \ . \end{aligned}$$

When f is strictly convex, it follows straightly from definition that the strict inequality sign in Jensen's inequality holds when $n = 2, x_1 \ne x_2$. In general, let us assume that the strictly inequality sign holds when $x_1, \cdots, x_{n-1}$ are distinct and prove it when $x_1, \cdots, x_n$ are not all equal. For, when all $x_1, \cdots, x_n$ are distinct, the second $\le$ in the above inequalities becomes $<$ due to the induction hypothesis and hence the strict inequality holds for n. When some x_j's are equal, we can group the expression $\sum_{j=1}^n \lambda_j x_j$ into $\sum_{j=1}^m \mu_j y_j$ where all y_j's are distinct and m is less than n. In this case the desired result comes from the induction hypothesis.

$\square$

When $\lambda_j \in [0,1]$, let $I_1 = \{j : \ \lambda_j \in (0,1]\}$ and $I_2 = \{j : \ \lambda_j = 0\}$. Then in the strictly convex case, equality sign holds if and only if $x_j = x_k$ for $j, k \in I_1$. The proof is essentially the same after observing that $\lambda_j x_j = 0$ and $\lambda_j f(x_j) = 0$ for $j \in I_2$ as well as $\sum_{j \in I_1} \lambda_j = 1$.

Jensen's inequality is applied to the strictly convex function e^x to yield

$$e^{\sum_{j=1}^n \lambda_j x_j} \le \sum_{j=1}^n \lambda_j e^{x_j} \ .$$

It can be rewritten as the generalized Young's inequality

$$a_1 a_2 \cdots a_n \le \frac{a_1^{p_1}}{p_1} + \frac{a_2^{p_2}}{p_2} + \cdots + \frac{a_n^{p_n}}{p_n}$$

where

$$a_j > 0,\ \sum_j \frac{1}{p_j} = 1,\ p_j > 1, j = 1, \cdots, n.$$

Moreover, the equality sign in this inequality holds if and only if all $a_j^{p_j}, j = 1, \cdots, n$, are equal. Taking and $x_j = a^{p_j}$ and $p_j = n$ for all j in the general Young's Inequality, we recover the AM-GM Inequality

$$(x_1 x_2 \cdots x_n)^{1/n} \le \frac{x_1 + x_2 + \cdots + x_n}{n}\ , \quad x_j > 0, j = 1, \cdots, n,$$

with equality holds if and only if all x_j's are equal. You may use the function $-\log x$ instead of e^x to obtain the same results. In the exercises other inequalities following from Jensen's are present.

Finally, we remark that in some books convexity is defined by a weaker condition, namely, a function f on I is convex if it satisfies

$$f\left(\frac{x+y}{2}\right) \le \frac{1}{2}\Big(f(x) + f(y)\Big), \quad \forall x, y \in I. \tag{1.5}$$

Indeed, this implies

$$f((1-\lambda)x + \lambda y) \le\ (1-\lambda) f(x) + \lambda f(y), \quad \forall x, y \in I,$$

provided f is continuous on I. I will leave it as an exercise. However, this conclusion does not hold without continuity. You may google under "weakly convex and continuity" for further information.